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A GEOMETRIC INTERPRETATION FOR THE TORSION CONSTRAINTS OF $(2,0)$ HETEROTIC WORLDSHEET SUPERGRAVITY

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ABSTRACT

We present a geometric interpretation for the torsion constraints in $(2,0)$ supergravity using G-structures. This leads to a classification of the constraints as given by [1]. We also present the essential torsion constraints for (p,q) geometry.

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1 Introduction

There exist different definitions of super Riemann surfaces in two dimensions.

i Patch definition [2, 3, 4]: A $N-$super Riemann surface can be defined as a collection of $\mathbf{C^{1|N}}$ supercoordinate neighbourhoods patched together by superconformal transformations. A coordinate transformation $(z, \theta^i) \rightarrow (\tilde{z}, \tilde{\theta}^i)$ is said to be superconformal if

$$D_i = F_i^{\,j} \tilde{D}_j \quad , \qquad F_i^{\,j} = D_i \tilde{\theta}^j \tag{1.1}$$

where $D_i = \partial_i + g_{ij}\theta^j \partial_z$ for $i = 1, \ldots, N$ and $g_{ij} = \delta_{ij}$.

ii Frame definition[4, 5, 6] A $N-$super Riemann surface is a complex manifold of dimension $1|N$ with a $0|N$ dimensional distribution $\mathcal{V}$ and the following extra structure. Let $\mathcal{T}$ be the tangent space of the manifold. Let E_α for $\alpha = 1, \ldots, N$ span $\mathcal{V}$ locally. Consider the symmetric bilinear form $B:\ \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{T}/\mathcal{V}$ given by

$$B(E_\alpha, E_\beta) \equiv [E_\alpha, E_\beta] \mod \mathcal{V} \quad , \tag{1.2}$$

where $E_\alpha, E_\beta \in \mathcal{V}$ and [,] is the graded Lie Bracket. The extra structure we impose is that the bilinear form B is non-degenerate.

The frame definition is coordinate invariant and hence leads to an intrinsic definition of super Riemann surfaces . It is also closely related to the supergravity formalism (wherein one introduces coordinate invariant objects by means of the vielbein). In supergravity, torsion constraints are introduced to fix the reducibility in the vielbein and connections. Their choice is rather mysterious from the supergravity point of view. It is almost an art rather than a science. The analysis of [1, 7] has provided a geometrical home for the torsion constraints of $N = 1$ supergravity. The torsion constraints (to be exact, some of them) were shown to correspond to a reduction of structure group. This led to a classification of the constraints into those which are essential and those which are inessential. The essential ones serve to impose a geometrical structure on the manifold. We shall provide a similar analysis for the $N = 2$ case. Some of these results have been obtained by [8]. This will naturally lead to the torsion constraints given in [9]. It can be shown that the supergravity(frame) definition of a $N = 2$ super Riemann surface is equivalent to the patch definition [5]. We then obtain the essential torsion constraints for (p, q) supergravity where p, q are positive integers.

2 Cartan's Theory of G-Structures

We shall briefly motivate G-Structures[10, 11, 7] by taking the simple example of obtaining a n-dim complex manifold from a $2n$-dim real manifold by means of G-structures. This example has been adapted from [7]. Any real manifold (supermanifold) M is endowed with a set of frames which span TM, the tangent space to the manifold. Let $\{E_A\}$ be the field of frames which span TM. A G-structure is imposed by defining two frames $\{E_A\}$, $\{E'_A\}$ to be equivalent to each other provided

$$E_A = G_A{}^B E'_B, \tag{2.3}$$

where G is specified depending on the structure one wants to impose on the manifold. The equivalence relation implies that the set of matrices that belong to G form a group. For the case in hand, the most general such G would correspond to an element of $GL(2n, \mathbf{R})$ which is the group of $2n \times 2n$ matrices with non-zero determinant. This corresponds to imposing no structure on the manifold. Additional structure is imposed by *reduction* of G [1]. An almost

[1]In general, all manifolds do not admit such a reduction. For example, it is well known that among all S^n, only S^2 and S^6 admit an "almost complex structure".

complex structure is imposed on M by restricting G to be of the form

$$G = \begin{pmatrix} R & 0 \\ 0 & R^* \end{pmatrix} \tag{2.4}$$

where we have chosen to work in a complex basis where the index A such that the first n correspond to the holomorphic coordinates z^i and the latter n to the antiholomorphic coordinates $\bar{z}^i$ and R are $n \times n$ complex matrices and R^* is the complex conjugate of R. It can be shown that this is the same as specifying the almost complex structure using the tensor J with $J^2 = -1$.

What does this have to do with torsion constraints? We define

$$[E_A, E_B] = t_{AB}{}^C E_C \quad . \tag{2.5}$$

The $t_{AB}{}^C$ are called structure constants. Further let $\hat{E}_A$ correspond to standard frame($\partial_{z^i}, \partial_{\bar{z}^i}$ for the example being studied.) and let $\hat{t}_{AB}{}^C$ be the corresponding structure constants(which are all zero for the n-dim complex manifold). Any arbitrary frame which is G-equivalent to $\hat{E}_A$ must be given by

$$E_A = G_A{}^B \hat{E}_B. \tag{2.6}$$

We then calculate $t_{AB}{}^C$ for the arbitrary frame. Certain $t_{AB}{}^C$ remain invariant for arbitrary G. For the case of complex structure, we find

$$t_{z^i z^j}{}^{\bar{z}^k} = 0 = t_{\bar{z}^i \bar{z}^j}{}^{z^k}. \tag{2.7}$$

The above equation is an example of a "torsion constraint" (One needs to do a little bit more work before it can be converted to a torsion constraint in supergravity. We shall get to it later.). By construction, the torsion constraints follow from the reduction of the structure group. Suppose we are given a frame which belongs to the G-structure, it is not always true that we can find a coordinate system where this frame is G-equivalent to the standard frame. We call a G-structure integrable if we can achieve this. It is easy to see that the torsion constraints are necessary conditions for the existence of such a coordinate system. It has been proved [12] that an almost complex structure is integrable provided the Nijenhuis tensor vanishes. The Nijenhuis tensor is given by

$$N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] \quad , \tag{2.8}$$

where X, Y are arbitrary vectors fields and J is the almost complex structure.

3 Torsion Constraints for $(2,0)$ supergravity

We shall now show that some of the torsion constraints of $(2,0)$ supergravity given in [9] can be obtained by the reduction of the structure group. We are given a smooth supermanifold with real dimension $2|4$. The standard frame is given by

$$\begin{aligned} \hat{E}_z &= \partial_z \\ \hat{E}_\theta &= \partial_\theta + \eta\, \partial_z \\ \hat{E}_\eta &= \partial_\eta + \theta\, \partial_z \end{aligned} \tag{3.9}$$

along with the complex conjugate equations. In the standard frame we have that all the structure constants are zero except $t_{\theta\eta}{}^{z} = 2$.

Complex Structure:

A complex structure corresponds to reduction of the structure group from $GL(2|4,\mathbf{R})$ to $GL(1|2,\mathbf{C})$. Explicitly, two frames are equivalent provided

$$\begin{pmatrix} E'_A \\ E'_{\bar{A}} \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & C^* \end{pmatrix} \begin{pmatrix} E_A \\ E_{\bar{A}} \end{pmatrix} \tag{3.10}$$

where C is a complex 3×3 matrix . and $A = z, \theta, \eta$ and $\bar{A} = \bar{z}, \bar{\theta}, \bar{\eta}$. We shall determine the "torsion constraints" which impose this complex structure by seeing which $t_{AB}{}^{C}$ remain unchanged under the action of G. We obtain

$$t_{A\ B}{}^{\bar{A}} = 0, \qquad t_{\bar{A}\ \bar{B}}{}^{A} = 0. \tag{3.11}$$

An "almost complex structure" on a 2|4 dimensional manifold corresponds to conditions (3.11).

Superconformal Structure:

Since we have already imposed a almost complex structure, we can restrict our attention to only the holomorphic sector. We have

$$\begin{pmatrix} E'_z \\ E'_\theta \\ E'_\eta \end{pmatrix} = \begin{pmatrix} C \end{pmatrix} \begin{pmatrix} E_z \\ E_\theta \\ E_\eta \end{pmatrix} \tag{3.12}$$

We define an "almost superconformal structure" by means of the reduction given by

$$\begin{aligned} \begin{pmatrix} E'_z \\ E'_\theta \\ E'_\eta \end{pmatrix} &= \begin{pmatrix} \hat{G} \end{pmatrix} \begin{pmatrix} E_z \\ E_\theta \\ E_\eta \end{pmatrix} \\ &\equiv \begin{pmatrix} a^2 & \vec{\Gamma} \\ \vec{0} & M \end{pmatrix} \begin{pmatrix} E_z \\ \vec{E} \end{pmatrix} \end{aligned} \tag{3.13}$$

where a is invertible, $\vec{\Gamma}$ is arbitrary and M is an invertible matrix. The $\vec{0}$ in (3.13) follows from $\vec{E}$ spanning a 0|2 dimensional distribution and the non-degeneracy condition implies that M is invertible. Now given an arbitrary frame in the G-structure defined by (3.13), it is convenient to go to frame where

$$B(E_\alpha, E_\beta) = 2g_{\alpha\beta}E_z \quad , \tag{3.14}$$

and E_z spans $\mathcal{T}/\mathcal{V}$. This leads to the "torsion constraint"

$$t_{\alpha\beta}{}^{z} = 2g_{\alpha\beta} \quad . \tag{3.15}$$

This is an example of a "torsion constraint" which we refer to as a gauge choice. Interestingly, unlike the case of almost complex structure, an almost superconformal structure does not lead to any essential torsion constraints. (3.14) is an inessential torsion constraint. Now the following subgroup $\tilde{G} \subset \hat{G}$ preserves the above gauge choice.

$$\begin{pmatrix} E'_z \\ E'_\theta \\ E'_\eta \end{pmatrix} = \begin{pmatrix} \tilde{G} \end{pmatrix} \begin{pmatrix} E_z \\ E_\theta \\ E_\eta \end{pmatrix} \equiv \begin{pmatrix} a^2 & \Gamma & \Lambda \\ 0 & ab & 0 \\ 0 & 0 & ab^{-1} \end{pmatrix} \begin{pmatrix} E_z \\ E_\theta \\ E_\eta \end{pmatrix} \tag{3.16}$$

where a, b are complex invertible even functions and Γ, Λ arbitrary odd functions. What "torsion constraints" correspond to this reduction? Again, as before we see which $t_{AB}{}^C$ are invariant under $\tilde{G}$. We find

$$t_{\bar{z}\alpha}{}^z = t_{\bar{z}\theta}{}^\eta = t_{\bar{z}\eta}{}^\theta = 0. \tag{3.17}$$

(3.11) and (3.17) lead to the so called "*essential* torsion constraints". They serve to impose complex and superconformal structure on the supermanifold. So far we have not made contact with the torsion constraints in supergravity. This is easily done by observing that[9]

$$T_{BC}{}^A = -t_{BC}{}^A + (\Phi_{BC}{}^A - (-)^{bc}\Phi_{CB}{}^A) + (A_{BC}{}^A - (-)^{bc}A_{CB}{}^A) \quad , \tag{3.18}$$

where Φ, A are the Lorentz and $U(1)_R$ connections respectively. This is obtained by identifying the vector field E_A with the differential operator $E_A = E_A{}^M \partial_M$. This is of the form

$$t_{BC}{}^A = -T_{BC}{}^A + \text{ terms involving connections.} \tag{3.19}$$

It can be shown that the "torsion constraints" in (3.11), (3.15) and (3.17) correspond to the case where the connection terms vanish in the above equation and hence can be immediately converted to real torsion constraints. They are

$$\begin{aligned} T_{A\ B}{}^{\bar{A}} = 0 \quad &, \qquad T_{\bar{A}\ \bar{B}}{}^{A} = 0, \\ T_{\alpha\beta}{}^{\ z} &= -2\delta_{\alpha\beta}, \\ T_{\bar{z}\alpha}{}^{z} = T_{\bar{z}\theta}{}^{\eta} &= T_{\bar{z}\eta}{}^{\theta} = 0. \end{aligned} \tag{3.20}$$

These give some of the torsion constraints chosen in [9]. The rest of the torsion constraints will be labelled *inessential*. They fall into two classes.

"Gauge" Choices.

"Conventional" constraints.

Under the action of arbitrary G, we have

$$\begin{aligned} \delta(2t_{\theta\eta}{}^{\theta} + t_{\theta\theta}{}^{\theta} + 2t_{\theta z}{}^{z}) &= -8\Lambda, \quad \text{and} \\ \delta(2t_{\eta\theta}{}^{\eta} + t_{\eta\eta}{}^{\eta} + 2t_{\eta z}{}^{z}) &= -8\Gamma \end{aligned} \tag{3.21}$$

We can now make the "gauge" choices

$$\begin{aligned} 2t_{\theta\eta}{}^{\theta} + t_{\theta\theta}{}^{\theta} + 2t_{\theta z}{}^{z} &= 0, \quad \text{and} \\ 2t_{\eta\theta}{}^{\eta} + t_{\eta\eta}{}^{\eta} + 2t_{\eta z}{}^{z} &= 0 \end{aligned} \tag{3.22}$$

and its complex conjugate equation by appropriately choosing Γ and Λ. It can be seen that this choice does not have an obstruction. One may however wonder why such a peculiar choice is made. The key point is that the peculiar combination is exactly the one required to remove all the connections from the relation involving components of torsion. This implies that this constraint can be easily converted to a real torsion constraint. We obtain

$$\begin{aligned} 2T_{\theta\eta}{}^{\theta} + T_{\theta\theta}{}^{\theta} + 2T_{\theta z}{}^{z} &= 0, \quad \text{and} \\ 2T_{\eta\theta}{}^{\eta} + T_{\eta\eta}{}^{\eta} + 2T_{\eta z}{}^{z} &= 0. \end{aligned} \tag{3.23}$$

where we implicitly assume the complex conjugate equations. For the rest of this section we will deal with the specific case of $(2,0)$ supergravity. Finally, the "conventional" constraints are those that serve to fix the Lorentz and $U(1)$ connections Φ and A. They are

Lorentz connection :

$$\begin{aligned} \phi_a &: \; T_{ab}{}^c = 0, \\ \phi_\alpha &: \; T_{\alpha\bar{z}}{}^{\bar{z}} = 0. \end{aligned} \tag{3.24}$$

$U(1)$ connection:

$$\begin{aligned} A_b &: \quad T_{b\theta}{}^\theta - T_{b\eta}{}^\eta = 0, \\ A_\theta &: \quad T_{\theta\theta}{}^\theta - 2T_{\theta\eta}{}^\eta = 0, \\ A_\eta &: \quad T_{\eta\eta}{}^\eta - 2T_{\eta\theta}{}^\theta = 0. \end{aligned} \tag{3.25}$$

This gives us all the torsion constraints of $(2,0)$ supergravity given in [9] after using the Bianchi identities.

We now derive the essential torsion constraints for the (p,q) geometry. The essential torsion constraints for (p,p) supergravity can be trivially obtained from the stucture group given by[2]

$$\begin{pmatrix} E'_z \\ E'_\alpha \end{pmatrix} = \begin{pmatrix} a^2 & * \\ 0 & aM_\alpha{}^\beta \end{pmatrix} \begin{pmatrix} E_z \\ E_\beta \end{pmatrix} \tag{3.26}$$

where a is invertible and M is an arbitrary $SO(p,\mathbf{C})$ matrix[4, 5] and $*$ refers to arbitrary odd elements. Here we have implicitly made the gauge choice given in (3.14). This leads to the torsion constraints.

$$\begin{aligned} T_{A\ B}{}^{\bar{A}} &= 0, \\ T_{\alpha\beta}{}^{z} &= -2\delta_{\alpha\beta}, \end{aligned}$$

[2]We assume that a complex structure has already been imposed and hence only give the holomorphic sector.

$$T_{\bar{z}\alpha}{}^{z} \quad = \quad 0, \tag{3.27}$$

and the complex conjugate equations. The inessential constraints are easily obtained case by case because they depend on the specific R-gauge groups one chooses to gauge. The torsion constraints for the (p, q)(with $q \leq p$) is obtained by truncating from the (p, p) case.

The integrability of the almost superconformal structure has been shown in [5]. This implies that given an arbitrary frame in the G-structure, we can always find a coordinate system where the frame is G-equivalent to the standard frame given in (3.9). It can be shown[5] that this implies that the two definitions of super Riemann surfaces earlier are equivalent.

4 Conclusion

In this paper, we have provided a geometric basis for the torsion constraints for the $(2, 0)$ supergravity and have also given the essential torsion constraints for the (p, q) case.

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