Ricci Flat metrics and the AdS-CFT correspondence

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Plan of talk

- Introduction
- Explicit construction of Ricci-Flat metrics
- The Leigh-Strassler deformation of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory
  - A conjecture on the spectrum of chiral primaries
  - Searching for the gravity dual to the CFT
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String Theory @ IITM: SG and Prasanta K. Tripathy

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Introduction
The AdS-CFT correspondence

In its original form, it relates type IIB strings propagating in a ten-dimensional spacetime $AdS_5 \times S^5$ to a four-dimensional conformal field theory (CFT): $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. [Maldacena 1997]

It is a strong-weak duality – strong coupling in string theory gets mapped to weakly coupled CFT and vice versa.

It makes it hard to verify but if true provides a nice way to carry out computations at strong coupling.
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- The duality that relates the Ising model at high-temperature to the Ising model a low-temperature is a similar example – it predicts a phase transition though one cannot obtain the critical temperature.
What is Ricci-Flatness?

A metric on a manifold assigns a length to any curve connecting a pair of points. Two nearby points with separation $dx^i$ ($i = 1, 2, \ldots, d$) are assigned a distance $ds$ defined by

$$ds^2 = g_{ij}(x) \, dx^i \, dx^j ,$$

We usually refer to $g_{ij}$ as the metric.

Is it possible to find coordinates such that $g_{ij} = \delta_{ij}$ everywhere? In general, the answer is no.

The best one can do in the neighbourhood of a point is a Taylor series of the form

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikjl} \, x^k \, x^l + O(x^3) .$$

The obstruction is called the Riemann curvature tensor.
What is Ricci-Flatness?

- The Ricci tensor is a second-rank symmetric tensor obtained from the Riemann curvature tensor by contracting a pair of indices $R_{ij} \equiv g^{kl} R_{ikjl}$.

- Einstein’s equations in general relativity is written in terms of this tensor ($R \equiv g^{ij} R_{ij}$)

$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi G_N T_{ij}.$$  

- A manifold is called Ricci-Flat (RF) if $R_{ij} = 0$ at all points.

- In 3 dimensions, the vanishing of the Ricci tensor implies the vanishing of the Riemann curvature tensor. This is not true in 4 and higher dimensions.
String theory requires spacetime to be ten-dimensional. However, spacetime as we perceive it at low energies is four-dimensional.

A simple and effective way to get around this is to assume that six of the dimensions are compact and small enough to be invisible at low energies.

String compactification assumes that spacetime is assumed to be of the form $\mathbb{R}^{1,3} \times M^6$, where $M$ is a compact six-dimensional manifold.

Consistency of string propagation (conformal invariance) requires $M$ to be Ricci-Flat to leading order.
AdS-CFT and Ricci-Flat manifolds

- Non-compact Ricci-Flat manifolds make an appearance in the context of the AdS-CFT correspondence (more generally, the gravity-gauge correspondence).

- This correspondence more generally relates type IIB string theory on a spacetime $\text{AdS}_5 \times X^5$ to a four-dimensional superconformal field theory (CFT).

- Let $M^6$ be a non-compact six-dimensional manifold obtained as a cone over $X^5$ i.e., consider a six-dimensional metric obtained from the five-dimensional metric on $X$ (which we write as $ds_X^2$):

$$ds_M^2 = dr^2 + r^2 \, ds_X^2, \quad r \in [0, \infty).$$

When $X = S^5$, $M = \mathbb{R}^6 = \mathbb{C}^3$.

- Consistency of string propagation on $\text{AdS}_5 \times X^5$ translates into the condition that $M$ be Ricci-Flat.
Consider the dynamical system, called the Ricci-Flow (due to Richard Hamilton)

\[ \frac{dg_{ij}}{dt} = -R_{ij}, \]

where $g_{ij}$ are the components of the metric.

Ricci-Flat metrics appear as the fixed-points of the dynamical system. This is an area being actively pursued in mathematics.

The proof of the Poincaré conjecture by Perelman makes use of this dynamical system.
Explicit Ricci-Flat metrics six-dimensional manifolds

with Aswin Balasubramanian and Chethan Gowdigere

*Symplectic potentials and resolved Ricci-Flat ACG metrics.*
Aswin K. Balasubramanian, SG, Chethan N. Gowdigere.

Kähler manifolds admit symplectic and complex structures that are compatible. For our purposes, it suffices to know that the metric is determined completely in terms of derivatives of a single function – schematically one has \( g_{ij} \sim \partial_i \partial_j G(x) \).

The condition of obtaining the RF metric reduces to solving a non-linear partial differential equation for the function. This is typically a hard problem.

In the context of non-linear DE’s such as the KdV equation, solutions have been found when there is an integrable structure.

Is there such a structure underlying finding RF Kähler manifolds?
Apostolov, Calderbank and Gauduchon (ACG) observed that if a Kähler manifold admits a Hamiltonian two-form, then there exists a special set of coordinates. In these coordinates, the metric takes a special form where the single function of several variables, $G(x)$ gets replaced by several functions of one variable.

This reduces the problem to solving several ODE’s. In fact, Ricci Flatness is easier to impose.

The neat result is that ACG provide a classification of metrics that admit a Hamiltonian two-form. In six-dimensions, such metrics are parametrised by an additional label, $\ell = 1, 2, 3$. Thus, there are three families of such metrics.
The \( \ell = 3 \) ACG metric

The special coordinates are \((\xi, \eta, \chi, t_1, t_2, t_3)\). The metric in these coordinates is (with \( \Delta = (\xi - \eta)(\eta - \chi)(\chi - \xi) \))

\[
\begin{align*}
ds^2 &= -\Delta \left[ \frac{d\xi^2}{(\eta - \chi) f(\xi)} + \frac{d\eta^2}{(\chi - \xi) g(\eta)} + \frac{d\chi^2}{(\xi - \eta) h(\chi)} \right] \\
&- \frac{1}{\Delta} \left[ (\eta - \chi) f(\xi) (dt_1 + (\eta + \chi) dt_2 + \eta \chi dt_3)^2 \\
&+ (\chi - \xi) g(\eta) (dt_1 + (\chi + \xi) dt_2 + \chi \xi dt_3)^2 \\
&+ (\xi - \eta) h(\chi) (dt_1 + (\xi + \eta) dt_2 + \xi \eta dt_3)^2 \right].
\end{align*}
\]
The $\ell = 3$ ACG metric

The special coordinates are $(\xi, \eta, \chi, t_1, t_2, t_3)$. The metric in these coordinates is (with $\Delta = (\xi - \eta) (\eta - \chi) (\chi - \xi)$)

\[ ds^2 = -\Delta \left[ \frac{d\xi^2}{(\eta - \chi) f(\xi)} + \frac{d\eta^2}{(\chi - \xi) g(\eta)} + \frac{d\chi^2}{(\xi - \eta) h(\chi)} \right] \\
- \frac{1}{\Delta} \left[ (\eta - \chi) f(\xi) (dt_1 + (\eta + \chi) dt_2 + \eta \chi dt_3)^2 \\
+ (\chi - \xi) g(\eta) (dt_1 + (\chi + \xi) dt_2 + \chi \xi dt_3)^2 \\
+ (\xi - \eta) h(\chi) (dt_1 + (\xi + \eta) dt_2 + \xi \eta dt_3)^2 \right]. \]

The scalar curvature is given by

\[ R = -\frac{f''(\xi)}{(\xi - \eta)(\xi - \chi)} - \frac{g''(\eta)}{(\eta - \xi)(\eta - \chi)} - \frac{h''(\chi)}{(\chi - \eta)(\chi - \xi)} \]
Our analysis

- We started with the three families of ACG metrics and carried out a global analysis by identifying the associated polytope.

- This global analysis enabled us to identify metrics for specific choices of the functions – it turns out that we always need $f/g/h$ to be cubic polynomials.

- All known examples of RF metrics appear in this class!

- In the $\ell = 1$ class, we obtained a new set of metrics. These correspond to a partial resolution of some well-known singular spaces, these are cones over five-dimensional spaces called $Y^{p,q}$ that appeared in the context of the AdS-CFT correspondence.

  [Gauntlett-Martelli-Sparks-Waldram]

- Our analysis is not exhaustive and there might be more!
The Leigh-Strassler deformation of $\mathcal{N} = 4$ SYM

(i) The spectrum of chiral primaries

with K. Madhu and Pramod Dominic

Chiral primaries in the Leigh-Strassler deformed $\mathcal{N}=4$ SYM - a perturbative study.
Kallingalthodi Madhu & SG

JHEP 05 (2007) 038. [hep-th/0703020]
The LS deformation of $\mathcal{N} = 4$ SYM

- The original AdS-CFT correspondence involves a CFT with a high degree of supersymmetry.
- Leigh and Strassler (LS) argued that this CFT admitted a two-parameter set of deformations that reduced supersymmetry to the minimal $\mathcal{N} = 1$.
- The LS theory has the same fields as in $\mathcal{N} = 4$ SYM theory. It contains one $\mathcal{N} = 1$ vector multiplet and three chiral multiplets that we will denote by $\Phi_1, \Phi_2, \Phi_3$, each of which transform in the adjoint of $SU(N)$ (not $U(N)$).
- The superpotential for $\mathcal{N} = 4$ SYM theory is

$$W_0 = h \ Tr \left( \Phi_1 \left[ \Phi_2, \Phi_3 \right] \right) = h \ Tr \left( \Phi_1 \Phi_2 \Phi_3 - \Phi_1 \Phi_3 \Phi_2 \right)$$
The superpotential for the LS theory is of the form

\[ W = W_0 + \frac{1}{3!} c^{ijk} \text{Tr} (\Phi_i \Phi_j \Phi_k), \]

where \( c^{ijk} \) is totally-symmetric in its indices.

It is useful to think of the three chiral fields as complex coordinates on \( \mathbb{C}^3 \).

\( c^{ijk} \) has 10 independent components, using simple linear redefinitions acting on the fields (\( SL(3, \mathbb{C}) \) acting on the three fields), we find only two non-trivial deformations. These are the two marginal deformations of Leigh and Strassler.
Chiral Primaries in CFT

- There exist a special class of operators in the CFT whose scaling dimension remains protected from quantum corrections (called ‘anomalous dimensions’).
- There exist a special class of operators with vanishing anomalous dimension that are called chiral primaries.
- Due to supersymmetry, several aspects of these operators at strong coupling can be studied as well. So it important to understand all such operators.
- We studied the spectrum of the (single-trace) chiral primaries by perturbatively computing the anomalous dimension of single-trace operators upto and including dimension six and looking for operators for which the anomalous dimension vanished.
The conjecture

Using a discrete non-abelian symmetry in theory given by the trihedral group $\Delta(27)$, we were able to classify the protected operators which lead to a sharp conjecture for arbitrary dimensions.

When the dimension $\Delta_0 > 2$, we conjectured [SG, Madhu]

<table>
<thead>
<tr>
<th>Scaling dim.</th>
<th>$\Delta_0 = 3r$</th>
<th>$\Delta_0 = a \mod 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{N} = 4$ theory</td>
<td>$\mathcal{L}<em>{0,0} \oplus \frac{r(r+1)}{2} \left[ \oplus</em>{i,j} \mathcal{L}_{i,j} \right]$</td>
<td>$(\Delta_0+1)(\Delta_0+2) \mathcal{V}_a$</td>
</tr>
<tr>
<td>$\beta$-def. theory</td>
<td>$\mathcal{L}<em>{0,0} \oplus j \mathcal{L}</em>{0,j}$</td>
<td>$\mathcal{V}_a$</td>
</tr>
<tr>
<td>LS theory</td>
<td>$2 \mathcal{L}_{0,0}$</td>
<td>$\mathcal{V}_a$</td>
</tr>
</tbody>
</table>

$\mathcal{L}_{i,j} (i, j = 0, 1, 2)$, $\mathcal{V}_1$ and $\mathcal{V}_2$ are irreps of $\Delta(27)$.

The conjecture was based on explicit computations up to and including $\Delta_0 = 6$. 
Towards proving the conjecture

- The conjecture has not been verified when $\Delta_0 > 6$ – direct calculations become very hard.

- At $\Delta = 6$, the computation involves 26 operators that mix quantum mechanically and the anomalous dimensions are given by the eigenvalues of a $26 \times 26$ matrix – this after using all symmetries else it would have been 58 dimensional.

- So we are pursuing a different approach. It has been shown that the one-loop anomalous dimensions of all chiral operators can be obtained as the spectrum of a one-dimensional supersymmetric spin-chain.

- The length of the spin-chain is mapped to $\Delta_0$. 
The spin-chain

- Due to supersymmetry, the energy eigenvalues of the Hamiltonian are bounded from below by zero.
- The vanishing of the anomalous dimensions gets mapped to a statement of number of eigenvectors that have eigenvalue zero.
- We are currently using the trihedral symmetry to organise the computation and we hope to prove the conjecture, at the very least, for spin-chains of length $3L$.

A much harder problem is to obtain the full spectrum of the spin-chain. Note that the spin chain has $3^N$ states where $N$ is the length of the chain.

- Is there a Bethe ansatz for this spin-chain? Yes, for the $\beta$-deformation.
Searching for the gravity dual to the LS theory

with Chethan Gowdigere

*Effective superpotentials for B-branes in Landau-Ginzburg models.*
SG and Hans Jockers.

The gravity dual for LS theory

While it is anticipated that the Leigh-Strassler theory will be dual to strings moving on a spacetime that is $AdS_5 \times X^5$ for some $X$, the precise space $X$ has not been determined!

Recall that the LS theory was obtained as a deformation of $\mathcal{N} = 4$ SYM theory for which $X = S^5$.

So the naive expectation is that $S^5$ should admit two deformations that are compatible with conformal invariance of string theory – i.e., a cone over $S^5$ which is nothing but $\mathbb{R}^6$ should admit deformations. There are none!

Is there a problem? However, the answer is known for a one-parameter deformation, the $\beta$-deformation.

[Lunin-Maldacena]
Generalising Ricci-Flatness

- In the context of string theory, the metric is one of several fields in theory.

- For instance, there is a second-rank antisymmetric tensor, called the B-field, \( B = \frac{1}{2} B_{ij} dx^i \wedge dx^j \), a scalar called the dilaton, \( \Phi \), that is common to all string theories.

- There are also other \( p \)-form gauge fields, \( C^{(p)} = \frac{1}{p!} C_{i_1 \ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \), that appear.

- So the condition for conformal invariance gives rise to a much more complicated system of coupled partial differential equations involving all these fields.
Generalising Ricci-Flatness

- In the limit that these fields vanish or take constant values, one recovers the condition that the metric must be Ricci-Flat.
- The term manifolds is typically used for spaces that are Riemannian and thus have a metric.
- A generalised manifold can be defined to be a space with the various massless fields of string theory.
- A generalised geometry on them is given by imposing the conditions for conformal invariance of string theory.
- The LS superpotential appeared in a computation of mine in a different context – this might give us a clue to finding the precise background.

[SG, Jockers]
Conclusion

I hope I have given you a flavour on some of the problems I am working on.
THANK YOU