# All about the Dirac delta function(?)

V. Balakrishnan

*Department of Physics*

*Indian Institute of Technology Madras, Chennai 600 036, India*

## Mouse games

As any child of ten will tell you, to write an article on the Dirac delta function (or on anything else, for that matter), one must first log into "Google" or "Yahoo" or a similar search engine. A judicious combination of clicking, cutting and pasting – and *voilá*, an article of any desired length is ready in an unbelievably short time!

Now this is not as simple as it sounds. It does require some finesse – possessed, no doubt, by the average ten-year-old (but not necessarily by older, less capable surfers). If one is naive enough to enter *delta function* in Google and click on search, the magician takes only 0.16 seconds to produce a staggering 1,100,000 possible references. Several lifetimes would not suffice to check all of these out. To make the search more meaningful, we enter "*delta function*" in quotes. This produces a less stupendous 58,600 references. As even this is too much, we try *Dirac delta function*, to get 52,500 references – not much of an improvement. Once again, "*Dirac delta function*" is much better, because Google then locates only 12,100 references. "*Dirac's delta function*" brings this down to 872, while "*the delta function of Dirac*" yields a comfortable (but not uniformly helpful) 19 references.

Motivated by a desire to include some interesting historical aspects in my article, I continued this fascinating pastime by trying *history of the Dirac delta function*, to be presented with 6,570 references to choose from. Spotting my mistake, I promptly moved to "*history of the Dirac delta function*", to be told that there were just 2 references, a most satisfying conclusion to the game. One of these was from *Mathematica*, and was as short and sweet as befits this impatient age. It said (in its entirety): O. Heaviside (1893-95), G. Kirchhoff (1891), P. A. M. Dirac (1926), L. Schwartz (1945). A true capsule history – provided you already knew the history! The other reference I didn't pursue, as the computer "froze" at this juncture. After all, the system had worked for nearly forty-five minutes without a hitch, and some such event was long overdue. The message was clear: it was time to get down to real work by shutting down the system and reverting to pencil and pad.

But what about the title of the article? Back to Google. Brief experimentation showed that "*All about the Dirac delta function*" produced *zero* references, so this title practically selected itself. Once this issue of *Reso-*

*nance* goes on-line, this article will be the *sole* reference, for the time being, if you cared to search under "*All about the Dirac delta function*" – but do note the all-important question mark in my title, added for the sake of truth and honesty!

## What the Dirac delta function looks like

Suppose $f(x)$ is a function that is defined, say, for all values of the real variable $x$, and that it is finite everywhere. Can we construct some sort of filter or "selector" that, when operating on this function, *singles out* the value of the function at any prescribed point $x_0$?

A hint is provided by the discrete analogue of this question. Suppose we have a sequence $(a_1, a_2, \ldots) = \{a_j | j = 1, 2, \ldots\}$. How do we select a particular member $a_i$ from the sequence? By summing over all members of (i.e., scanning!) the sequence with a selector called the Kronecker delta, denoted by $\delta_{ij}$ and defined as $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. It follows immediately that

$$\sum_{j=1} \delta_{ij}\, a_j = a_i\,. \tag{1}$$

Further, we have the normalization $\sum_j \delta_{ij} = 1$ for each value of $i$, and also the symmetry property $\delta_{ij} = \delta_{ji}$. Reverting to the continuous case, we must replace the summation over $j$ by an integration over $x$. The role of the specified index $i$ is played by the specified point $x_0$. The analogue of the Kronecker delta is written like a function, retaining the same symbol $\delta$ for it. (Presumably, this was Dirac's reason for choosing this notation for the delta function.) So we seek a "function" $\delta(x - x_0)$ such that

$$\int_{-\infty}^{\infty} dx\, \delta(x - x_0)\, f(x) = f(x_0)\,. \tag{2}$$

Exactly as in the discrete case of the Kronecker delta, we impose the normalization and symmetry properties

$$\int_{-\infty}^{\infty} dx\, \delta(x - x_0) = 1 \quad \text{and} \quad \delta(x - x_0) = \delta(x_0 - x)\,. \tag{3}$$

The requirements in Eqs. (2) and (3) may be taken to define the Dirac delta function. The form of Eq. (2) suggests that $\delta(x - x_0)$ is more like the kernel of an integral *operator* than a conventional function. We will return to this aspect subsequently.

What can $\delta(x - x_0)$ possibly look like? A naive way of answering this question is as follows. Take a rectangular window of width $2\varepsilon$ and height

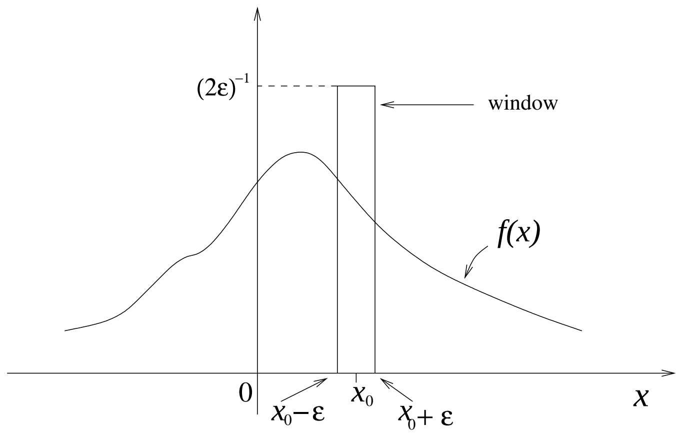


Figure 1: The window reduces to the delta function when $\varepsilon \to 0$

$1/(2\varepsilon)$, so that the area of the window is unity. Place it with its bottom edge on the $x$ axis and slide it along this axis, as shown in Fig. 1. When the window is centred at the chosen point $x_0$, the integral of $f(x)$ multiplied by this window function is simply $(1/2\varepsilon)\int_{x_0-\varepsilon}^{x_0+\varepsilon} dx\, f(x)$ . This does not quite select $f(x_0)$ alone, of course. But it will do so if we take the limit $\varepsilon \to 0$. In this limit, the width of the window becomes vanishingly small. Simultaneously, its height becomes arbitrarily large, so as to "capture" all of the ordinate in the graph of $f(x)$, no matter how large the value of $f(x_0)$ is. A possible explicit form for the Dirac delta function $\delta(x - x_0)$ is therefore given by

$$\delta(x - x_0) = \begin{cases} \lim_{\varepsilon\to 0} 1/(2\varepsilon)\,, & \text{for } x_0 - \varepsilon < x < x_0 + \varepsilon \\ 0\,, & \text{for all other } x\,. \end{cases} \tag{4}$$

This cannot be a stand-alone definition. If it is taken literally, then, formally, $\delta(x - x_0)$ must be zero for all $x \neq x_0$, while it must be infinite for $x = x_0$ . An explicit form such as Eq. (4) for the delta function must be interpreted in the light of Eq. (2). The delta "function" is always to be understood as something that makes sense when it occurs in an integral like Eq. (2), i.e., when it acts on ordinary functions like $f(x)$ and an integration is carried out. It is immediately clear that the so-called Dirac delta "function" cannot be a function in the conventional sense. In particular, $\delta(x - x_0)$ must be *singular* (formally infinite) at $x = x_0$, that is, at the point where its argument is zero.

Mathematically, an explicit form for the Dirac delta function is properly given in terms of a *sequence* or family of conventional functions, rather than the "window" representation in Eq. (4). It can then be arranged that, in a suitable limit, the sequence approaches a quantity that has all the properties desired of the delta function. An infinite number of such sequences may be constructed. For instance, take a family of functions $\phi_\varepsilon(x - x_0)$ parametrized by a positive constant $\varepsilon$, and with the following properties: each member of the family (i) has a peak at $x_0$, (ii) is symmetric about the point $x_0$, and (iii) has a definite integral from $-\infty$ to $\infty$ whose value is unity. Matters are arranged such that, as the parameter $\varepsilon$ is made smaller and smaller, the height of the peak in $\phi_\varepsilon(x)$ increases while its width simultaneously decreases, keeping the total area under the curve equal to unity. Then $\lim_{\varepsilon\to 0} \phi_\varepsilon(x - x_0)$ represents the delta function $\delta(x - x_0)$. Let us now write down the simplest choices for such sequences. For ease of writing, let us set $x_0 = 0$. One of the simplest possibilities is the family of "Lorentzians", given by

$$\phi_\varepsilon(x) = \frac{\varepsilon}{\pi(x^2 + \varepsilon^2)}\,. \tag{5}$$

Then $\lim_{\varepsilon\to 0} \phi_\varepsilon(x)$ is a representation of the Dirac delta function $\delta(x)$, with the properties specified in Eqs. (2) and (3). Some other popular choices for $\phi_\varepsilon(x)$ are the following:

$$\frac{1}{2\varepsilon}\,\exp\left(-|x|/\varepsilon\right); \quad \frac{1}{2\sqrt{\pi\varepsilon}}\,\exp\left(-x^2/4\varepsilon\right); \quad \frac{\text{sech}^2(x/\varepsilon)}{2\varepsilon}; \quad \frac{\sin(x/\varepsilon)}{\pi x}; \ldots \tag{6}$$

It is instructive to sketch these functions schematically, and to check out what happens as smaller and smaller values of $\varepsilon$ are chosen. As an amusing exercise, think up your own sequence of functions that leads to the delta function as a limiting case.

What's the point of all this? Before going on to answer this question, it's helpful to re-write the last of the functions in (6) as follows. If we put $\varepsilon = 1/L$, we get

$$\delta(x) = \lim_{L\to\infty} \frac{\sin(L\,x)}{\pi\,x} = \frac{1}{2\pi}\lim_{L\to\infty}\int_{-L}^{L} dk\, e^{ikx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\,. \tag{7}$$

This turns out to be perhaps the most useful way of representing the delta function. Since $|e^{ikx}| = 1$, it is obvious that the last integral in Eq. (7) is not absolutely convergent. Nor is the integral well-defined in the ordinary sense, because sin $kx$ and cos $kx$ do not have definite limits as $k \to \pm\infty$. These are just further reminders of the fact that the delta function is not a conventional function, as we have already emphasized. Those young readers who are familiar with Fourier transforms will recognize that the last equation above seems to suggest that the Fourier transform of the Dirac delta function is just unity. This is indeed so. It suggests, too, that one way of *defining* "singular" functions like the delta function might be *via* their Fourier transforms: for example, we could define $\delta(x)$ as the inverse Fourier transform of a constant – in this case, just unity.

## Some history

The Dirac delta function has quite a fascinating history. The book by Lützen, cited at the end of this article, is an excellent source of information. The delta function seems to have made its first appearance in the early part of the $19^{\text{th}}$ century, in the works of Poisson (1815), Fourier (1822), and Cauchy (1823, 1827). Poisson and Cauchy essentially used arguments that implied that the Lorentzian representation of the delta function, Eq. (5), had the "selector property" stated in Eq. (2). Fourier, in his fundamental work *Théorie Analytique de la Chaleur*, showed (in connection with Fourier series expansions of periodic functions) that the series $1/(2\pi) + (1/\pi)\sum_{n=1}^{\infty} \cos n(x - x_0)$ had precisely this sort of selector property, i.e., was a representation of $\delta(x - x_0)$ in the fundamental interval $(x - x_0) \in [-\pi, \pi]$. His arguments essentially amount to the last of the representations in Eq. (6) for $\delta(x)$. These early works did not aim at mathematical rigour in the current sense of the term. Subsequently, Kirchhoff (1882, 1891) and Heaviside (1893, 1899) gave the first mathematical definitions (again non-rigorous, by modern standards) of the delta function. Kirchhoff was concerned with the fundamental solution of the three-dimensional wave equation, while Heaviside introduced the function in his "Operational Calculus". He pointed out that $\delta(x)$ could be regarded as the derivative of the Heaviside or unit "step function" $\theta(x)$, defined as unity for $x > 0$ and zero for $x < 0$. After Heaviside, the delta function was freely used – in particular, in connection with Laplace transforms, especially by electrical engineers (e.g., Van der Pol, 1928). Dirac (1926, 1930) introduced it in his classic and fundamental work on quantum mechanics, essentially as the continuous analogue of the Kronecker delta. He also wrote down a list of its important properties – much the same list that standard textbooks now carry. Over and above Eqs. (2) and (3), the delta function also satisfies

$$x\,\delta(x) = 0\,, \quad \delta\,'(-x) = -\delta\,'(x)\,, \quad x\,\delta\,'(x) = -\delta(x)\,, \quad \delta(ax) = (1/|a|)\,\delta(x) \tag{8}$$

where $a$ is any real number, and so on. Again, these equations are to be understood as valid when multiplied by suitable smooth functions and integrated over $x$. Dirac also listed the useful but not immediately obvious property

$$\delta(x^2 - x_0^2) = \frac{\delta(x + x_0) + \delta(x - x_0)}{2|x_0|} \tag{9}$$

where $x_0$ is any real number.

The use of the delta function became more and more common after the appearance of Dirac's work. Other singular functions also made their appearance, as early versions of quantum field theory began to take shape in the works of physicists such as Jordan, Pauli and Heisenberg. Around the same time, mathematicians began attempts to define such singular quantities in a rigorous manner. The delta function and other such singular objects were recognized to be what are called *generalized functions* or *distributions*, rather than functions in the conventional sense. The first rigorous theory was given by Bochner in 1932. Soon afterwards, Sobolev (1935) gave the rigorous definition of distributions as *functionals*, and the way had been paved for a definitive mathematical theory. This was achieved by Schwartz (1945-50), and comprehensively treated in his *Théorie des Distributions*, Vol. 1 (1950) and Vol. 2 (1951). For lack of space, we will not go further into these aspects, other than to repeat that we now have a completely rigorous mathematical theory of distributions.

## Why does the $\delta$-function appear in physical problems?

We can now turn to the question of why the delta function appears so naturally in physical problems. Consider, for example, the basic problem of electrostatics: given a static charge density $\rho(\mathbf{r})$ in free space, what is the corresponding electrostatic potential $\phi(\mathbf{r})$ at any arbitrary point $\mathbf{r} = (x, y, z)$? From Maxwell's equations, we know that $\phi$ satisfies Poisson's equation, namely,

$$\nabla^2 \phi(\mathbf{r}) = -\rho(\mathbf{r})/\epsilon_0 \,, \tag{10}$$

where $\epsilon_0$ is the permittivity of the vacuum. What does one do in the case of a *point* charge $q$ located at some point $\mathbf{r}_0 = (x_0\,, y_0\,, z_0)$? A point charge is an idealization in which a *finite* amount of charge $q$ is supposed to be packed into zero volume. The charge density must therefore be infinite at the point $\mathbf{r}_0$ , and zero elsewhere. The delta function comes to our aid: we may write, in this case,

$$\rho(\mathbf{r}) = q\, \delta(x - x_0)\, \delta(y - y_0)\, \delta(z - z_0) \equiv q\, \delta^{(3)}(\mathbf{r} - \mathbf{r}_0)\,, \tag{11}$$

where the *three-dimensional* delta function $\delta^{(3)}$ is short-hand for the product of the three delta functions in the equation above. It is easy to verify that this expression for $\rho(\mathbf{r})$ has all the properties required of a point charge at the point $\mathbf{r}_0$ . This illustrates how (and why) the delta function frequently appears as the right-hand side of fundamental equations of mathematical physics. It turns out that it also appears as the singular part of fundamental *solutions* to basic equations such as the wave equation.

It is worth noting that representations of "higher-dimensional" delta functions like $\delta^{(3)}$ are easily written down. For instance, the three-dimensional counterpart of Eq. (7) above is just

$$\delta^{(3)}(\mathbf{r}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} dk_3\, e^{i\,(k_1x+k_2y+k_3z)} \equiv \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\, e^{i\,\mathbf{k}\cdot\mathbf{r}}\,. \tag{12}$$

The notation used in the final equation should be self-explanatory.

We have mentioned earlier that Poisson was responsible for what was perhaps the first recognizable use of the Dirac delta function. Poisson and Dirac seem to be linked in more ways than one. The most profound of these links is this: Dirac showed that Poisson brackets in classical dynamics become the commutators of the corresponding operators in quantum mechanics, multiplied by the constant factor $2\pi/ih$ where $h$ stands for Planck's constant. It is therefore appropriate to end this short account with another fascinating link between the names of Dirac and Poisson. There is a very useful and remarkable result in Fourier analysis called the Poisson summation formula. In its simplest form, this says that if $\widetilde{f}(k)$ is the Fourier transform of $f(x)$, then

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widetilde{f}(2\pi n)\,. \tag{13}$$

A very elegant and simple way of deriving this formula makes use of the so-called "Dirac comb" : an array of Dirac delta functions located at the integers. It can be shown that

$$\sum_{n=-\infty}^{\infty} \delta(x-n) = \sum_{n=-\infty}^{\infty} e^{2\pi n i x}\,, \tag{14}$$

i.e., the Dirac comb is identically equal to a sum of exponentials! The latter can be reduced to the expression $1 + 2\sum_1^{\infty}\cos(2\pi n x)$. The cosines in the sum "interfere destructively" with each other, leaving behind just the sharp $\delta$-function spikes at integer values of $x$. With the help of Eq. (14), the Poisson summation formula is established quite easily. We have thus come full circle, moving from Poisson to Dirac and returning to Poisson with the help of Dirac.

## Suggested Reading

1. P. A. M. Dirac, *The physical interpretation of the* [sic] *quantum mechanics*, Proc. Roy. Soc. A **113**, 621-641 (1926). So heady was the progress in that incredible period that the definite article preceding "quantum mechanics" was presumably no longer needed by 1930! P.A.M. Dirac,

*The Principles of Quantum Mechanics*, Oxford University Press, Oxford, 1930 (original edition). An inexpensive Indian reprint of the latest edition is available.

2. M. J. Lighthill, *Introduction to Fourier Analysis and Generalized Functions*, Cambridge University Press, Cambridge. Originally published in 1958, this little classic has been reprinted several times. The book's dedication is as succinct as its text, and says: "To Paul Dirac, who saw that it must be true, Laurent Schwartz, who proved it, and George Temple, who showed how simple it could be made".

3. J. Lützen, *The Prehistory of the Theory of Distributions*, Springer-Verlag, New York, 1982.

# Box

## Why did Dirac need the delta function?

The delta function appeared in Dirac's work on quantum mechanics in an *avatar* somewhat different from the ones mentioned in the text.

Consider ordinary three-dimensional (Euclidean) space. This is a *linear vector space* (LVS). Any vector in it can be expanded uniquely as a linear combination of the three unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. This is because these three vectors are *linearly independent* of each other, and they *span* the space: i.e., they form a *basis* in the LVS. Moreover, $\mathbf{i}\cdot\mathbf{i} = 1$, $\mathbf{i}\cdot\mathbf{j} = 0$, etc. Using the superior notation $\mathbf{e}_1 \equiv \mathbf{i}$, $\mathbf{e}_2 \equiv \mathbf{j}$ and $\mathbf{e}_3 \equiv \mathbf{k}$, all these relations can be compressed into $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ $(i, j = 1,\ 2 \text{ or } 3)$, in terms of the Kronecker delta. That is, the set $\{\mathbf{e}_i \,|\, i = 1, 2, 3\}$ is an *orthonormal* basis. This can be generalized to any $n$-dimensional LVS: an orthonormal basis $\{\mathbf{e}_i\}$ satisfying $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ $(i, j = 1,\ 2, \ldots, n)$ can always be chosen in it.

What happens if the dimensionality $n \to \infty$? Some subtleties arise. But the preceding discussion goes through, provided care is taken to ensure that certain desirable properties survive – e.g., the vectors in the LVS must have finite magnitudes, and the triangle inequality must be satisfied by the magnitude of the resultant of two vectors. *Function spaces* provide simple examples of such infinite-dimensional LVS's – for instance, the space of all square-integrable functions of $x$ in some interval $[a, b]$. Naturally, the basis is then an infinite set of suitable functions. A common example is the set of Legendre polynomials $\{P_\ell(x)\}$ where $x \in [-1, 1]$ and $\ell = 0, 1, \ldots$ *ad inf.* The notion of the dot product of two vectors must also be generalized appropriately. We do not go into this detail here.

But a new possibility arises when the dimensionality of an LVS is infinite. It may have a basis that is itself *uncountably* infinite, i.e., a so-called *continuous* basis. Instead of a set $\{\mathbf{e}_i\}$ where $i$ is a discrete index, we have a set $\{\mathbf{e}(\xi)\}$ where $\xi$ is a continuous variable, taking values in some range. (For simplicity we continue to use the symbol $\mathbf{e}$ for the basis elements of the LVS, even when these may be functions or other objects.) Omitting several technical details, the orthonormality condition for a continuous basis formally reads $\mathbf{e}(\xi)\cdot\mathbf{e}(\xi\,') = \delta(\xi - \xi\,')$. That is, the Dirac delta function replaces the Kronecker delta.

This is the context in which Dirac required the delta function. In quantum mechanics, a system is described by its so-called state vector. This is an element of a certain LVS called the Hilbert space of the system. The classical dynamical observables of the system are replaced by operators that act on the elements of its Hilbert space. It turns out to be convenient to choose the eigenstates of (a subset of) these physical operators as the possible basis sets in the Hilbert space. Moreover, the system gets into these eigenstates when measurements of the corresponding physical observables are made. Certain fundamental operators such as the position operator or linear momentum operator of a particle moving in a given force field turn out to have continuous sets of eigenvalues. Their eigenstates then constitute continuous basis sets, with the orthonormality condition as given by the last equation above.

An example is provided by the case of a particle moving in one dimension under the influence of a constant force. Its energy $E$ can then be shown to have a continuous set of possible values. The state $\mathbf{e}(E)$ of the particle corresponding to a definite value $E$ of its total energy is a member of a continuous basis set of states, satisfying $\mathbf{e}(E) \cdot \mathbf{e}(E') = \delta(E - E')$.