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Claudio G. Carvalhaes and Patrick Suppes

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Approximations for the period of the simple pendulum based on the arithmetic-geometric mean

Claudio G. Carvalhaes
Center for the Study of Language and Information, Ventura Hall, 210 Panama Street, Stanford University, Stanford, California, 94305-4101 and Instituto de Matemática e Estatística, Universidade do Estado do Rio de Janeiro, Rio de Janeiro, RJ, 20550-900, Brazil

Patrick Suppes
Center for the Study of Language and Information, Ventura Hall, 210 Panama Street, Stanford University, Stanford, California, 94305-4101

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We use the arithmetic-geometric mean to derive approximate solutions for the period of the simple pendulum. The fast convergence of the arithmetic-geometric mean yields accurate solutions. We also discuss the invention of the pendulum clock by Christiaan Huygens in 1656–1657. © 2008 American Association of Physics Teachers.

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I. INTRODUCTION

The period of the simple pendulum can be expressed as a nonelementary function of its amplitude and cannot be evaluated analytically without approximation. The linearization of the equation of motion gives an approximate solution for small oscillations. In this regime the pendulum behaves like a harmonic oscillator and is a classic example of isochronism. For larger oscillations the pendulum’s motion is anisochronous, and the linear equation does not describe the oscillations. The lack of an elementary closed-form restricts the study of the simple pendulum at the undergraduate level to small oscillations. Approaches based on the expansion of the period in terms of a power series of the amplitude are sometimes presented in advanced courses, but converge slowly even for intermediate amplitudes. Several alternatives have been proposed that differ in complexity and domains of validity; some are intuitive, and others are based on relatively sophisticated procedures.

In this paper we study a sequence of approximations for the pendulum period based on the use of the arithmetic-geometric mean. This quantity has been used to approximate the complete elliptic integrals of the first and the second kind with high accuracy. The idea is to compute a sequence of arithmetic and geometric means of two positive numbers which converge toward each other as more iterations are performed. The convergence is proportional to the reciprocal of the complete elliptic integral of the first kind, and so the sequence can be used to accurately determine the pendulum period for arbitrary amplitudes. For instance, convergence with eight digits of accuracy is obtained for an amplitude of 179° after only four iterations.

If the arithmetic-geometric mean’s iterations are calculated analytically, the successive means provide a sequence of analytical approximations for the period. The relation between two successive iterations of this sequence has a clear physics interpretation in terms of a renormalization process in which a given pendulum is replaced by another of the same period but of smaller amplitude, until the iteration reaches a sufficiently small amplitude that the calculation of the period is accurately handled by a simple approximation.

Section II reviews the formulation of the period of the nonlinear pendulum in terms of the complete elliptic integral of the first kind. In Sec. III we define the arithmetic-geometric mean, illustrate its calculation, and show how it is related to the pendulum period. We also highlight the numerical constraints on the computation of the exact period. In Sec. IV we present approximate solutions for the period based on the arithmetic-geometric mean and compare them to other published solutions. After our conclusions in Sec. V, we include in Sec. VI a historical note on the invention of the pendulum clock by Christiaan Huygens.

II. THE PENDULUM PERIOD

A simple pendulum consists of a particle of mass m hanging from an unstretchable, rigid massless rod of length L fixed at a pivot point. The system freely oscillates in a vertical plane under the action of gravity. It is assumed that the motion is not affected by damping or external forcing. For the initial conditions \( \theta(0)=\theta_0 \) and \( d\theta/dt(0)=0 \), with the zero of the potential energy at the bottom of the pendulum’s trajectory as shown in Fig. 1, the equation of energy conservation is

\[
mgL(1-\cos \theta) = \frac{1}{2}mL^2 \left( \frac{d\theta}{dt} \right)^2 + mgL(1-\cos \theta),
\]

where \( g \) is the local acceleration of gravity. We solve Eq. (1) for the angular velocity \( d\theta/dt \) in the first quarter of its period and obtain

\[
T = T_0 \sqrt{\frac{2}{\theta}} \int_0^\theta \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}},
\]

where \( T_0 = 2\pi \sqrt{L/g} \) is the period in the small-angle approximation. The improper integral in Eq. (2) is cast into a more convenient form by using the identity \( \cos \theta = 1 - 2 \sin^2(\theta/2) \) and the change of variable \( \sin \phi = \sin(\theta/2)/\sin(\theta_0/2) \), which leads to

\[
T = \frac{2}{\pi} K(k) T_0,
\]

where \( k = \sin(\theta_0/2) \), and

\[
K(k) = \int_0^\pi \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}}.
\]
and hence,14,17,18 authors21 consider it to correspond to the largest interval for $T$. The range in which the particle is displaced to the angle $\theta_0$ and then released at rest. The system oscillates under the action of gravity. The angular displacement $\theta$ is measured counter-clockwise relative to the equilibrium position ($\theta=0$).

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad (|k| < 1 \text{ and } |\theta_0| < 180^\circ).$$

(4)

The integral in Eq. (4) is called a complete elliptic integral of the first kind. For small amplitudes ($k \approx 0$), $K$ converges to $\pi/2$ so that the amplitude-independent approximation $T_0$ readily emerges from Eq. (3). As shown in Fig. 2, $K$ is a monotonically increasing function, so that $T_0$ underestimates $T$ with increasing inaccuracy. The range in which $T_0$ is a valid approximation for $T$ is somewhat arbitrary, but some authors21 consider it to correspond to the largest interval for which $T_0$ differs from the exact value of $T$ by less than 1%, which is roughly $|\theta_0| < 23^\circ$.

III. THE ARITHMETIC-GEOMETRIC MEAN

The function $K(k)$ cannot be expressed in terms of elementary functions, but can be numerically evaluated with high precision by a simple procedure. To do so we first introduce the two-term recursion relation

$$a_n = \frac{1}{2}(a_{n-1} + b_{n-1}), \quad (5a)$$

$$b_n = \sqrt{a_{n-1}b_{n-1}}, \quad (5b)$$

where $a$ and $b$ are real numbers such that $0 \leq b < a$ with $a_0 = a$ and $b_0 = b$. It can be shown that $\sqrt{a_n} \geq \sqrt{b_n}$ for all $n \geq 1$, and hence,\textsuperscript{14,17,18}

$$a_n - b_n \leq \frac{1}{2}(a_{n-1} - b_{n-1}) \leq \cdots \leq \frac{1}{2^n}(a - b). \quad (6)$$

Because the right-hand side of this inequality goes to zero as $n \to \infty$, the sequences $a_n$ and $b_n$ converge to a common limit uniquely determined by $a_0$ and $b_0$. This limit is denoted $M(a,b)$ and is called the arithmetic-geometric mean of $a$ and $b$.

$$M(a,b) = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n. \quad (7)$$

Table I illustrates the solution of the recursion relations (5) for $(a,b)=(1,0.8)$. Agreement of 15 decimal places between the means is obtained after only four iterations, suggesting that the arithmetic-geometric mean rapidly converges. Because the convergence of the arithmetic-geometric means is quadratic, an agreement of about $2^n$ digits between the means is expected after $n$ iterations. To see so we introduce a measure of the error at the $n$th iteration

$$e_n = \sqrt{a_n^2 - b_n^2}. \quad (8)$$

We observe that

$$e_{n+1} = \frac{1}{2}(a_n - b_n) = \frac{e_n^2}{4a_{n+1}}. \quad (9)$$

We see that the sequence $e_n$ goes to 0 quadratically, so that the convergence of the arithmetic-geometric means is second order (quadratic).

We introduce the family of integrals

$$I(a,b) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}}, \quad (a,b > 0), \quad (10)$$

which contains $K$ as the special case

$$K(k) = I(1,\sqrt{1-k^2}). \quad (11)$$

The change of variable $t=\gamma \tan \phi$ followed by $u=(t-\sqrt{a} t)/\sqrt{2}$ and some rearrangements in Eq. (11) leads to the key relation

$$I(a,b) = I(a_1,b_1), \quad (12)$$

where $a_1$ and $b_1$ are the arithmetic and geometric means of $a$ and $b$, respectively. Equation (12) shows that $I$ is invariant under the mapping $(a,b) \to (a_1,b_1)$. Successive applications of this property yield

$$I(a,b) = I(a_n,b_n), \quad (13)$$

and $I$ is invariant under the arithmetic-geometric mean recursion relation in Eq. (5). If we take the limit $n \to \infty$ in Eq. (13)
and use the continuity of \( I(a, b) \) and the definition Eq. (7), we obtain
\[
I(a, b) = \lim_{n \to \infty} I(a_n, b_n) = I(M(a, b), M(a, b)) = \int_0^{\pi/2} \frac{d\phi}{M(a, b)} = \frac{\pi}{2M(a, b)}.
\]

(14)

It follows from Eq. (11) that
\[
K(k) = \frac{\pi}{2M(1, \sqrt{1-k^2})} = \frac{\pi}{2a_\infty},
\]

(15)

where \( a_\infty = \lim_{n \to \infty} a_n \). We use this result in Eq. (3) to infer that
\[
T = \frac{T_0}{a_\infty}.
\]

(16)

Equation (16) yields the exact solution of the pendulum period in terms of the arithmetic-geometric mean. It also represents an iterative algorithm for determining \( T \) numerically for arbitrary amplitudes. The only constraint on this computation is the finite representation of real numbers in a computer. Our computations used the IEEE floating-point standard, which defines the distance from 1.0 to the next largest 64-bit double format number as \( \varepsilon = 2^{-52} \). This number is called the machine epsilon or the unit round-off. For this criterion the iterations cease at the first step for which the error \( e_n \) given by Eq. (8) satisfies \( e_n < \varepsilon \). The number of iterations need to achieve this precision depends on the amplitude. The quadratic convergence of the arithmetic-geometric means ensures that only a few iterations are necessary, even for very large amplitudes. This statement is confirmed by Table II, which shows that only seven iterations are sufficient to obtain the exact solution to machine precision for a wide range of amplitudes. For the interval \( |\theta_0| < 179^\circ \) the number of required iterations reduces to 3 to achieve agreement within 1% with the exact value. The computation of \( T \) in this way is easily performed by using a spreadsheet or a pocket calculator.

### IV. ARITHMETIC-GEOMETRIC-MEAN BASED APPROXIMATIONS FOR THE PENDULUM PERIOD

If the calculations of the recursion relations of Eq. (16) are performed analytically, a sequence \( \{T_n\} \) of algebraic expressions is obtained which converges to the exact solution in the ranges shown in Table II. The elements of \( \{T_n\} \) are given by
\[
T_n = \frac{T_0}{a_n},
\]

(17)

where \( a_n \) is the arithmetic mean at the \( n \)th iteration in the calculation of \( M(1, \sqrt{1-k^2}) \). Let \( q = \sqrt{1-k^2} = \cos(\theta_0/2) \). Then the first four iterations give

\[
T_1 = \frac{2T_0}{1 + q},
\]

(18a)

\[
T_2 = \frac{4T_0}{1 + q + 2q^{1/2}},
\]

(18b)

\[
T_3 = \frac{8T_0}{1 + q + 2q^{1/2} + 2^{3/2}q^{1/4}(1 + q)^{1/2}},
\]

(18c)

\[
T_4 = \frac{16T_0}{1 + q + 2q^{1/2} + 2^{3/2}q^{1/4}(1 + q)^{1/2} + 2^{7/4}q^{1/8}(1 + q)^{1/4}(1 + q + 2q^{1/2})^{1/2}}.
\]

(18d)

The approximation \( T_n \approx T_0 \) for small amplitudes is recovered from Eq. (18) in the limit \( q \to 1 \) (\( \theta_0 = 0 \)). The approximation \( T_0(1 + \theta_0^2/16) \), often cited as the first correction to \( T_0 \), emerges from the expansion of \( T_1 \) in \( \theta_0 \), truncated at first order. For the upper limit \( \theta_0 = 180^\circ \) (\( q = 0 \)) we obtain the sequence \( \{2T_0, 4T_0, 8T_0, 16T_0, \ldots\} = \{2^nT_0\} \), which shows the exponential divergence of the exact solution \( T_\infty \).

Equation (18) represent a good approximation for the period. From Table II we see that seven iterations were needed to achieve a result for the period for \( |\theta_0| < 179.99^\circ \) with machine epsilon precision. Further computations were performed with the aid of a symbolic manipulation package. Because the analytical expressions for \( T_5, T_6, \) and \( T_7 \) are too long, they are have not given here. The expression for \( T_{n+1} \) can be obtained from \( T_n \) by replacing \( q \) by \( 2\sqrt{q/(1+q)} \) (the harmonic mean of 1 and \( q \)) and replacing \( T_0 \) by \( 2T_0/(1+q) \),...
Table III. Approximate expressions for the period of the simple pendulum given by various authors. $T_0$ was taken equal to one and $A_1$ and $A_2$ are as in Table II.

<table>
<thead>
<tr>
<th>Reference</th>
<th>Approximation</th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$1 + \theta_0/16$</td>
<td>0.03°</td>
<td>73.91°</td>
</tr>
<tr>
<td>5</td>
<td>$[(\sqrt{3} \theta_0/2)/\sin(\sqrt{3} \theta_0/2)] - 1$</td>
<td>0.08°</td>
<td>135.07°</td>
</tr>
<tr>
<td>6</td>
<td>$1 - \theta_0^2/8$</td>
<td>0.03°</td>
<td>78.50°</td>
</tr>
<tr>
<td>7</td>
<td>$(\theta_0/\sin (\theta_0))^{3/4}$</td>
<td>0.05°</td>
<td>111.87°</td>
</tr>
<tr>
<td>8</td>
<td>$1/\sqrt{q}$</td>
<td>0.04°</td>
<td>95.79°</td>
</tr>
<tr>
<td>11</td>
<td>$(\sqrt{q})^{1/2}$</td>
<td>1.11°</td>
<td>177.15°</td>
</tr>
<tr>
<td>12</td>
<td>$[\theta_0/2J(\theta_0)]^{-2}$</td>
<td>0.05°</td>
<td>116.56°</td>
</tr>
<tr>
<td>13</td>
<td>$\log q/(q-1)$</td>
<td>0.05°</td>
<td>120.54°</td>
</tr>
</tbody>
</table>

Equation (19) provides an alternative procedure for computing the arithmetic-geometric mean approximations recursively and is suitable for computer algebra implementations. It also allows us to establish the arithmetic-geometric mean as a recursive replacement by another of smaller amplitude but of the same period. Let $q_{n+1} = 2q_n/(1+q_n)$ and $T_{n+1} = 2T_n/(1+q_n)$, with $(q_1, T_1) = (q, T_0)$.

For convenience, we associate each pair $(q_n, T_n)$ with an effective amplitude $\theta_n$ and an effective length $L_n$ to obtain

$$T_n = T_1(\theta_n, L_n),$$

where $q_n = \cos \theta_n/2$, $T_n = 2\pi \sqrt{L_n/g}$, and $(\theta_1, L_1) = (\theta_0, L)$. It follows from the inequalities $q_0 > q_{n-1} > \cdots > q$ and $T_0 > T_{n-1} > \cdots > T_n$ that $\theta_0 < \theta_1 < \cdots < \theta_n$ and $L_0 > L_1 > \cdots > L_n$.

At the end of the treatise, Huygens reaches a result equivalent to the result for small angles $T = 2\pi \sqrt{L/g}$.

Huygens had no knowledge of the arithmetic-geometric mean, which was first analyzed in a paper by Lagrange (1784–85), and much studied by Gauss as a young man in unpublished papers, the one on arithmetic-geometric mean having been written in 1799 or 1800.

VI. HISTORICAL NOTE

The pendulum clock was invented by Christiaan Huygens in the winter of 1656–1657. In 1673 Huygens published his magnum opus. It is now generally recognized that, after Newton’s 1687 Principia, Huygens’ treatise was the most original work in mathematical physics in the 17th century. An early anticipation of the principle of conservation of energy is stated as the single new Hypothesis or Axiom required for determining the center of oscillation.

If any number of weights begin to move by the force of their own gravity, their center of gravity cannot rise higher than the place at which it was located at the beginning of the quote (Ref. 28, p. 108).

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aElectronic mail: claudioc@stanford.edu
bElectronic mail: psuppes@stanford.edu
dS. Thornton and J. Marion, Classical Dynamics of Particles and Systems, 4th ed. (Saunders College, Fort Worth, TX, 1995).


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