Central forces

For any central force the equation of motion is

\[ V_{\text{eff}}(r) = V(r) + \frac{L^2}{2mr^2} \]

\[ \frac{d^2u}{d\varphi^2} + u = -\frac{m}{L^2} \frac{1}{u^2} f\left(\frac{1}{u}\right) \]

Motion in inverse square law force field

\[ f(r) = \frac{k}{r^2} \]

\[ V_{\text{eff}} = \frac{k}{r} + \frac{L^2}{2mr^2} \]

\[ E = \frac{1}{2} m\dot{r}^2 + \frac{k}{r} + \frac{L^2}{2mr^2} \]

\[ f_{\text{eff}}(r) = \frac{k}{r^2} + \frac{L^2}{mr^3} \]

\[ \frac{1}{r} = A \cos(\varphi - \varphi_0) - \frac{mk}{L^2} \]
inverse square law forces

Plot of effective potential

\[ V_{eff} = \frac{k}{r} + \frac{L^2}{2mr^2} \]
inverse square law forces

Plot of effective potential

\[ V_{\text{eff}} = \frac{k}{r} + \frac{L^2}{2mr^2} \]

- \( k/r \)
- \( L^2/ (2mr^2) \)
- \( E_1 \)
- \( E_2 = 0 \)
- \( E_3 \)
- \( E_4 \)
Central forces

\[ \frac{1}{r} = A \cos(\varphi - \varphi_0) - \frac{mk}{L^2} \]

\[ r = \frac{1}{A \cos \varphi - (mk / L^2)} = r_0 \frac{1+e}{1+e \cos \varphi} \]

where ‘e’ is the eccentricity of the orbit

- e < 1, the orbit is an ellipse
- e = 0, the orbit is a circle
- e = 1, the orbit is a parabola
- e > 1, the orbit is a hyperbola
review

✓ **Central force**
  conservative; planar orbits;
  constants of motion: $L$, $A$ and $E$

✓ **Inverse square force**
  orbits: conic sections

✓ **Orbits for different values of $E$**
  shapes determined by $L$ and $E$
  parameters: $A$, $e$, $E$, $r_0$, $m$ and $k$

✓ **Geometry of the conic sections**
  ellipse, parabola, hyperbola
inverse square law orbits

\[ e = \sqrt{1 + \frac{2EL^2}{mk^2}} \]

**special cases:**

(i) \( E > 0, \ e > 1 \)  
hyperbola

(ii) \( E = 0, \ e = 1 \),  
\( r_2 \) becomes infinity  
parabola

(iii) \( E < 0, \ e < 1 \)  
two turning points,  
ellipse

(iv) \( E = -mk^2 / (2L^2), \ e = 0 \),  
circle